

# A Note on the Liouville Equation

A. Grauel

Fachbereich 6 der Universität-Gesamthochschule Paderborn

Z. Naturforsch. **36a**, 417–418 (1981);  
received March 18, 1981

We study some geometrical features of the non-linear scattering equations [1]. From this we deduce the Liouville equation. For that we interpret the  $SL(2, \mathbb{R})$ -valued elements of the matrices in the scattering equations as matrix-valued forms and calculate the curvature 2-form with respect to a basis of the Lie algebra. We obtain the Liouville equation if the curvature form is equal to zero.

We give a geometrical interpretation for the nonlinear evolution equation, namely the Liouville equation. To that let us start with the scattering problem in the form

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}_{,x} = \begin{pmatrix} \eta & q(x, t) \\ r(x, t) & -\eta \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}. \quad (1)$$

The time evolution of the functions  $\varphi^1(x, t)$  and  $\varphi^2(x, t)$  is given by

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}_{,t} = \begin{pmatrix} A(x, t; \eta) & B(x, t; \eta) \\ C(x, t; \eta) & -A(x, t; \eta) \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad (2)$$

where  $\varphi^i_{,x} = \partial\varphi^i/\partial x$ ,  $\varphi^i_{,t} = \partial\varphi^i/\partial t$  and  $i = 1, 2$ . The quantity  $\eta$  is called eigenvalue of the scattering problem and the quantities  $q(x, t)$ ,  $r(x, t)$ ,  $A(x, t; \eta)$ ,  $B(x, t; \eta)$  and  $C(x, t; \eta)$  must be given to specify the specific problem. If we rewrite (1) and (2) in matrix notation then we obtain

$$\varphi^k_{,j} + \sum_p \Gamma^k_{pj} \varphi^p = 0, \quad (3)$$

where  $j, k, p, q = 1, 2$  and  $x^1 = x$ ,  $x^2 = t$  and  $\varphi^j(x, t)$  are interpreted as the components of a two-component field on the principal bundle  $P = P(M, G)$ . The  $\Gamma^k_{qj}$  are given by the components of the matrix in (1) and (2).

The curvature form [2] is given by the exterior covariant derivative of the 1-form  $\omega$  on  $P$  with values in a finite-dimensional vector space  $V$  in the form

$$\Omega = \nabla\omega = d\omega \circ h, \quad (4)$$

Reprint requests to Dr. A. Grauel, Fachbereich 6, Universität-Gesamthochschule Paderborn, Warburger Straße 100, D-4790 Paderborn.

where  $\Omega$  is a  $\mathfrak{g}$ -valued 2-form and

$$\begin{aligned} \nabla\omega(X_1, \dots, X_{p+1}) \\ = d\omega(hX_1, \dots, hX_{p+1}), \end{aligned} \quad (5)$$

where  $h: T_p(P(M, G)) \rightarrow S_p$  the projection of the tangential space  $T_p = S_p \otimes V_p$  onto its horizontal subspace  $S_p$ . The space  $V_p$  of vertical vectors lies tangential to the fibre.

The exterior derivative  $d$  is unchanged in its action on forms which take their values in a real vector space  $V$ . On sections of

$$V \otimes \Lambda^1\{T_p(P(M, G))\}$$

we have

$$d(X_j \otimes \omega^j) = X_j \otimes d\omega^j, \quad \omega^j \in \Lambda^1(T_p), \quad (6)$$

where  $\{X_k\}_{k=1}^n$  is a basis for  $V$ . If  $V = \mathfrak{g}$  we can write

$$\begin{aligned} [X_i \otimes \omega^i, X_j \otimes \omega^j] \\ = (\omega^i \wedge \omega^j) \otimes [X_i, X_j], \end{aligned} \quad (7)$$

where we have related  $\mathbb{R}$ -valued forms to the bracket of  $\mathfrak{g}$ -valued forms. Equation (7) is anticommutative and satisfies the Jacobi identity. Now we are in a position to express the curvature form (4). Let  $\{X_k\}_{k=1}^3$  be a basis of the Lie algebra  $\mathfrak{g} = SL(2, \mathbb{R})$ , then with (6) and (7) we obtain the curvature form

$$\begin{aligned} \Omega = \sum_{i=1}^3 d\omega^i \otimes X_i \\ + \frac{1}{2} \sum_{i,j=1}^3 (\omega^i \wedge \omega^j) \otimes [X_i, X_j], \end{aligned} \quad (8)$$

where  $\omega^k$  ( $k = 1, 2, 3$ ) are arbitrary 1-forms and  $[X_p, X_q]$  is the commutator of the quantities  $X_k$ . We choose

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (9)$$

as a basis of  $\mathfrak{g}$ . In view of (3) we can write for the 1-forms

$$\begin{aligned} \omega^1 &= -(\eta dx + A dt), \\ \omega^2 &= -(q dx + B dt), \\ \omega^3 &= -(\eta dx + C dt), \end{aligned} \quad (10)$$

If we take into account (9) and (10), then we can give the curvature form (8) in the explicit expression

0340-4811 / 81 / 0400-0417 \$ 01.00/0. — Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

$$\begin{aligned}\Omega = & (qC - rB - A_x) dx \wedge dt \otimes X_1 \\ & + (2\eta B - 2qA + q_t - B_x) dx \wedge dt \otimes X_2 \\ & + (-2\eta C + 2rA + r_t - C_x) dx \wedge dt \otimes X_3,\end{aligned}\quad (11)$$

where  $\eta \neq \eta(t)$ . The explicit expression (11) is now applied to the Liouville equation. The coefficients  $A$ ,  $B$  and  $C$  in (2), (11) are one-parameter families of functions of  $x$ ,  $t$  and  $q$ ,  $r$  with their derivatives. The parameter is the quantity  $\eta$ . We choose

$$\begin{aligned}A &= -\frac{1}{4\eta} \cosh u - \frac{1}{4\eta} \sinh u, \\ B &= -C = -\frac{1}{4\eta} \sinh u - \frac{1}{4\eta} \cosh u, \\ r &= q = -\frac{u_x}{2},\end{aligned}\quad (12)$$

and obtain

$$\Omega = -\frac{1}{2} (u_{xt} + e^u) dx \wedge dt \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (13)$$

If

$$\Omega = 0, \quad (14)$$

we have

$$u_{xt} + e^u = 0, \quad (15)$$

the Liouville equation. Moreover, from condition (14) we conclude that

- i)  $\omega$  satisfies the Maurer-Cartan structural equation  $d\omega + \frac{1}{2}[\omega, \omega] = 0$ ,
- ii) the connection in  $P(M, G)$  is flat.

*Final Remark:* We have given a geometrical interpretation of a physically important example, namely the Liouville equation. The geometrical consideration states that the Liouville equation is contained in the scattering equations. Moreover we see that  $\omega$  satisfies the structure equation of Maurer-Cartan. The Maurer-Cartan equation implies that the canonical flat connection has zero curvature [2]. The existence of pseudopotentials is considered in [3], furthermore the fact that the Liouville equation cannot be solved by inverse scattering methods.

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Letters **31**, 125 (1973).

[2] S. Kobayashi and K. Nomizu, Foundation of Differential Geometry, Vol. I, Interscience Publishers, London 1963.

[3] R. Sasaki, Phys. Letters **73A**, 77 (1979).