A Note on the Liouville Equation

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We study some geometrical features of the non-linear scattering equations [1]. From this we deduce the Liouville equation. For that we interpret the SL(2, ℝ)-valued elements of the matrices in the scattering equations as matrix-valued forms and calculate the curvature 2-form with respect to a basis of the Lie algebra. We obtain the Liouville equation if the curvature form is equal to zero.

We give a geometrical interpretation for the nonlinear evolution equation, namely the Liouville equation. To that let us start with the scattering problem in the form

$$\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}_{,x} = \begin{pmatrix}\eta & q(x,t)\\ r(x,t) & -\eta\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}. \qquad (1)$$

The time evolution of the functions $\varphi^1(x,t)$ and $\varphi^2(x,t)$ is given by

$$\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}_{,t} = \begin{pmatrix}A(x,t;\eta) & B(x,t;\eta)\\ C(x,t;\eta) & -A(x,t;\eta)\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}, \qquad (2)$$

where $\varphi^i_{,x} = \partial\varphi^i/\partial x$, $\varphi^i_{,t} = \partial\varphi^i/\partial t$ and $i = 1,2$. The quantity η is called eigenvalue of the scattering problem and the quantities $q(x,t)$, $r(x,t)$, $A(x,t;\eta)$, $B(x,t;\eta)$ and $C(x,t;\eta)$ must be given to specify the specific problem. If we rewrite (1) and (2) in matrix notation then we obtain

$$\varphi^k_{,j} + \sum_p \Gamma^k_{pj}\varphi^p = 0, \qquad (3)$$

where $j, k, p, q = 1,2$ and $x^1 = x$, $x^2 = t$ and $\varphi^j(x,t)$ are interpreted as the components of a two-component field on the principal bundle $P = P(M, G)$. The Γ^k_{qj} are given by the components of the matrix in (1) and (2).

The curvature form [2] is given by the exterior covariant derivative of the 1-form ω on P with values in a finite-dimensional vector space V in the form

$$\Omega = \nabla\omega = \mathrm{d}\omega \circ h, \qquad (4)$$

where Ω is a $\mathfrak{g}$-valued 2-form and

$$\nabla\omega(X_1, \ldots, X_{p+1}) = \mathrm{d}\omega(hX_1, \ldots, hX_{p+1}), \qquad (5)$$

where $h\colon T_p(P(M,G)) \to S_p$ the projection of the tangential space $T_p = S_p \otimes V_p$ onto its horizontal subspace S_p. The space V_p of vertical vectors lies tangential to the fibre.

The exterior derivative d is unchanged in its action on forms which take their values in a real vector space V. On sections of

$$V \otimes \Lambda^1\{T_p(P(M,G))\}$$

we have

$$d(X_j \otimes \omega^j) = X_j \otimes \mathrm{d}\omega^j, \quad \omega^j \in \Lambda^1(T_p), \qquad (6)$$

where $\{X_k\}_{k=1}^n$ is a basis for V. If $V = \mathfrak{g}$ we can write

$$[X_i \otimes \omega^i, X_j \otimes \omega^j] = (\omega^i \wedge \omega^j) \otimes [X_i, X_j], \qquad (7)$$

where we have related ℝ-valued forms to the bracket of $\mathfrak{g}$-valued forms. Equation (7) is anticommutative and satisfies the Jacobi identity. Now we are in a position to express the curvature form (4). Let $\{X_k\}_{k=1}^3$ be a basis of the Lie algebra $\mathfrak{g} = \mathrm{SL}(2, \mathbb{R})$, then with (6) and (7) we obtain the curvature form

$$\Omega = \sum_{i=1}^{3} \mathrm{d}\omega^i \otimes X_i + \tfrac{1}{2}\sum_{i,j=1}^{3} (\omega^i \wedge \omega^j) \otimes [X_i, X_j], \qquad (8)$$

where $\omega^k\,(k = 1,2,3)$ are arbitrary 1-forms and $[X_p, X_q]$ is the commutator of the quantities X_k. We choose

$$X_1 = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}, \quad X_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad X_3 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad (9)$$

as a basis of $\mathfrak{g}$. In view of (3) we can write for the 1-forms

$$\omega^1 = -(\eta\,\mathrm{d}x + A\,\mathrm{d}t),$$
$$\omega^2 = -(q\,\mathrm{d}x + B\,\mathrm{d}t),$$
$$\omega^3 = -(\eta\,\mathrm{d}x + C\,\mathrm{d}t), \qquad (10)$$

If we take into account (9) and (10), then we can give the curvature form (8) in the explicit expression

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$$\Omega = (qC - rB - A_x)\,\mathrm{d}x \wedge \mathrm{d}t \otimes X_1 + (2\eta B - 2qA + q_t - B_x)\,\mathrm{d}x \wedge \mathrm{d}t \otimes X_2 + (-2\eta C + 2rA + r_t - C_x)\,\mathrm{d}x \wedge \mathrm{d}t \otimes X_3, \tag{11}$$

where $\eta \neq \eta(t)$. The explicit expression (11) is now applied to the Liouville equation. The coefficients A, B and C in (2), (11) are one-parameter families of functions of x, t and q, r with their derivatives. The parameter is the quantity η. We choose

$$A = -\frac{1}{4\eta}\cosh u - \frac{1}{4\eta}\sinh u, \quad B = -C = -\frac{1}{4\eta}\sinh u - \frac{1}{4\eta}\cosh u, \quad r = q = -\frac{u_x}{2}, \tag{12}$$

and obtain

$$\Omega = -\frac{1}{2}(u_{xt} + e^u)\,\mathrm{d}x \wedge \mathrm{d}t \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

If

$$\Omega = 0, \tag{14}$$

we have

$$u_{xt} + e^u = 0, \tag{15}$$

the Liouville equation. Moreover, from condition (14) we conclude that

i) ω satisfies the Maurer-Cartan structural equation $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0$,

ii) the connection in $P(M, G)$ is flat.

Final Remark: We have given a geometrical interpretation of a physically important example, namely the Liouville equation. The geometrical consideration states that the Liouville equation is contained in the scattering equations. Moreover we see that ω satisfies the structure equation of Maurer-Cartan. The Maurer-Cartan equation implies that the canonical flat connection has zero curvature [2]. The existence of pseudopotentials is considered in [3], furthermore the fact that the Liouville equation cannot be solved by inverse scattering methods.

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[3] R. Sasaki, Phys. Letters **73 A**, 77 (1979).